which resembles a treatment in the old strong-coupling theory.

Another possibility of extension is to search for a larger group which includes the SU_6 group and the Lorentz group as subgroups. One possible group of this kind is the SL_6 group (unimodula linear-transformation group in a six-dimensional complex-vector space), which contains $SL_2 \otimes SL_3$ as a subgroup. However, it seems to be impossible to have a wave equation compatible with this group without extending the four-dimensional Minkowsky space to a higher dimensional space (36-dimensional space!).[16]

[16] After this paper was written, the author became aware of a paper by F. Gürsey and L. A. Radicati [Phys. Rev. Letters 13, 173 (1964)], in which they claim that the relativistic extension of the theory is possible.

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Self-Consistent Calculation for the Nucleon Mass, π-N Coupling Constant, and the Low-Energy $P_{1/2}$, $T=\frac{1}{2}$, π-N Scattering Phase Shifts

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A self-consistent calculation is performed for the low-energy $P_{1/2}$, $T=\frac{1}{2}$, pion-nucleon scattering amplitude, using the N/D method. The distant part of the left-hand cut for the partial-wave amplitude is dealt with in the manner of Balázs. The mass of the nucleon bound state is found to be ≈ 880 MeV, the coupling constant ≈ 12. The low-energy phase shifts and the scattering length are also in good qualitative agreement with the observed ones.

I. INTRODUCTION

DURING the past few years, the low-energy pion-nucleon system has been subjected to extensive investigations within the framework of dispersion theory. The main feature of such investigations has been to calculate the pion-nucleon scattering amplitude (to be precise, the various low angular-momentum partial-wave amplitudes) given by forces that arise from the low-mass intermediate states in the crossed $\pi-N$ and $\pi\pi \to N\bar{N}$ channels.[1,2] These low-mass states are the familiar nucleon and the $P_{3/2}$, $T=\frac{3}{2}$, $\pi-N$ resonance, N^* for the crossed $\pi-N$ channel and the $T=1$, $J=1$, $\pi\pi$ resonance, i.e., the ρ meson, for the channel $\pi\pi \to N\bar{N}$. For a more accurate determination of the $\pi-N$ scattering amplitude, one must of course incorporate in the problem the forces that arise from the higher mass intermediate states in the crossed channels. In the language of the N/D method, that provides the appropriate technique for such calculations, this amounts to taking into account the contributions that arise from the distant part of the left-hand cut of the various partial-wave amplitudes in the s plane. This, however, is a difficult problem. A method to tackle it has been suggested by Balázs,[3] wherein these far left-hand cut contributions can be approximated by a set of pole terms; the locations of these poles are known but the residues are, as such, unknown constants. However, if one knows the amplitude correctly at any point in the low-energy region, these residues can be determined by matching, at this point, the expression for the amplitude involving these residues with the known amplitude.[4] The latter quantity can be obtained from the absorptive parts of the crossed channels through a fixed-energy dispersion relation.

The above method, if reliable, would certainly lead to a more accurate determination of any scattering amplitude than if the far left-hand cut contributions were just ignored, but one would still have to treat the crossed channels as known. For instance, if the various low-energy, low angular-momentum partial-wave amplitudes for $\pi-N$ scattering were calculated and one thus found a bound state (nucleon) and a resonance (N^*), it would be only after one inserted them beforehand in the crossed $\pi-N$ channel. At this stage, however, if the criterion of self-consistency[5,6] is invoked,

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[1] S. C. Frautschi and J. D. Walecka, Phys. Rev. 120, 1486 (1960). We follow the notation of this paper; note that our units are $\hbar=c=m_\pi=1$.

[2] J. S. Ball and D. Wong, Phys. Rev. 133, B179 (1964).

[3] L. A. P. Balazs, Phys. Rev. 128, 1939 (1962).

[4] One will have to match the amplitude and the first $n-1$ derivatives, if the number of Balázs residues to be determined is n.

[5] G. F. Chew, *Proceedings of the International Conference on High-energy Nuclear Physics, Geneva, 1962*, edited by J. Prentki (CERN, Geneva, 1962), p. 525.

[6] G. F. Chew, Phys. Rev. Letters 9, 233 (1962).

and found compatible with the results of the calculation—by which we mean if the crossed channel $\pi-N$ forces arising from N and N^* can be described in terms of certain parameters, the numerical values of which coincide with those of N and N^* obtained in our calculation—we will have explained these objects without having to really assume them. It is of course believed that the amplitude that the calculation yields is unique.

In the present calculation, we have tried to study the nucleon in the above spirit. Actually, it would have been more desirable to calculate both N and N^* at one stroke, as one knows that the nucleon provides the main force to create N^*, and N^* provides the main force to bind the nucleon.[6] However, we shall see that in the present approach, given the N^*, the calculation of N itself becomes a self-consistency problem. This is what happens in the case of N^* also, which has been dealt with by Singh and Udgaonkar.[7] The present calculation can be taken to be complementary to theirs.

Before going into the details of the nucleon problem, we would, however, like to make a comment on the possible pitfalls that are associated with the matching criterion that one needs to invoke in the calculations of the present type. It seems that the final results would depend crucially on the choice of the point at which one matches the N/D amplitude with the one obtained from the fixed-energy dispersion relation. This certainly seems to happen for the $\pi-\pi$ problem.[8] In the present calculation, although the region where one could "match" has to be chosen quite carefully, as discussed at the end of this paper, yet one has a finite region from which one could choose the matching point. We find that there certainly is variation in our results as we vary the matching point in this region. However, it is gratifying to note that the particular matching point corresponding to which we have done detailed calculations for the nucleon problem, i.e., for the $P_{1/2}$, $T=\frac{1}{2}$ partial-wave amplitude, leads to fairly satisfactory results for this as well as for several other low angular-momentum partial waves.[7,9,10] This makes the present approach quite worthwhile, in the sense that although one probably has the above "preferred" matching point as a parameter, it suffices to enable one to calculate all low angular-momentum partial waves. Of course it is true that the meaning of the parameter is at the moment obscure.[11]

It should be mentioned that since, in the present calculation, the nucleon mass turns out to be lower than the actual one, the calculated $\pi-N$ threshold is lower than the physical one. The physical threshold of course means the point $s=(m+1)^2$ with $m\approx 6.8$—the experimental nucleon mass; the calculated threshold corresponds to $m\approx 6.3$, which is the mass we obtain for the nucleon. Hence, there is some ambiguity as to whether the scattering length obtained at the calculated threshold should be a more valid quantity for comparison with the experimental scattering length. We have calculated the scattering length at both thresholds, but have laid emphasis only on a qualitative comparison.

The self-consistent value of the coupling constant obtained by us is in fair agreement with the observed one, but it is smaller than the latter. In the light of a calculation by Abers and Zemach,[12] who get $g^2\approx 19$, this at first sight seems a little curious. However, this difference is not quite unexpected, as the above authors ignore the forces other than those arising from N, N^*, and ρ exchange (i.e., the forces coming from the distant part of the left-hand cut), whereas we do not. Another point that should be kept in mind is that if one obtains a bound state arising principally from the force due to the exchange of this bound state itself, the coupling constant one calculates is the same as the one that occurs in the force term. In that case, if one finds that the calculated bound state is more tightly bound than is observed, the force responsible for this binding should also exceed the actual force. Consequently, the calculated coupling constant should turn out to be larger than the observed one. The situation we encounter is, however, not the same, as the main force for the binding of the nucleon comes, not from the exchange of the nucleon itself, but from the exchange of N^*.

II. DETAILS

The kinematical details of the problem are well known. Using the notation of Ref. 1, we choose our amplitude as

$$g_{11}(s)\equiv g_{1-}^{(T=1/2)}(s)=(W^2/q^3)e^{i\delta_{11}}\sin\delta_{11}. \qquad (1)$$

We have

$$g_{11}(s)=(1/32\pi q^2)\{[(W+m)^2-1][A_1+(W-m)B_1] +[(W-m)^2-1][-A_0+(W-m)B_0]\}, \qquad (2)$$

where A_1, B_1, and A_0, B_0 are the $l=1$ and $l=0$ partial-wave projections of the $T=\frac{1}{2}$ combinations of the invariant amplitudes $A^{\pm}(s,t,u)$ and $B^{\pm}(s,t,u)$.

We can express $g_{11}(s)$ as N/D, with D having the unitarity cut and N incorporating in it all other singularities of the amplitude $g_{11}(s)$ (Fig. 1):

$$g_{11}(s)=N(s)/D(s)\,. \qquad (3)$$

$D(s)$, after one subtraction, can be expressed as

$$D(s)=1-\frac{(s-s_0)}{\pi}\int_{(m+1)^2}^{\infty}\frac{(q'^3/s')N(s')}{(s'-s)(s'-s_0)}ds'. \qquad (4)$$

$s=W^2$ is, as usual, the square of the center-of-mass energy for the "direct" $\pi-N$ channel. We shall work in

[7] V. Singh and B. M. Udgaonkar, Phys. Rev. 130, 1177 (1963).

[8] M. L. Mehta and P. K. Srivastava, University of Delhi, 1964 (to be published).

[9] S. K. Bose and S. N. Biswas, 134, B635 (1964).

[10] P. Narayanaswamy and L. K. Pande, Nuovo Cimento 33, 468 (1964).

[11] Note that this parameter has certainly nothing to do with the cutoff parameter that one needs in calculations of bound states and resonances in other approaches, e.g., Ref. 2 and 12.

[12] E. Abers and C. Zemach, Phys. Rev. 131, 2305 (1963).

the s plane throughout.[13] The relevant (approximate) analytic structure of $N(s)$ is displayed by the relation

$$N(s)=\frac{b_1}{s-s_1}+\frac{b_2}{s-s_2}+\frac{1}{\pi}\int_{-\infty}^{0} ds'\frac{D(s')\,\mathrm{Im}g_{11}(s')}{s'-s}\,, \tag{5}$$

where the first two terms are the approximate expressions for the shortcuts arising from the crossed channel N^* and N respectively. The residues b_1 and b_2 are of course known[1,6,14]:

$$b_1=(32/9)s_1^{3/2}\gamma_{33}D(s_1)\,,$$
$$b_2=\tfrac{2}{3}f^2m^3D(m^2)\,.$$

The remaining part of the N function is approximated, following the method of Balázs, by a two-pole expression:

$$b_3/(s-s_3)+b_4/(s-s_4)\,, \tag{6}$$

where[7] $s_3=-m^2$, $s_4=-16m^2$ and b_3 and b_4 are two unknown residues. Thus we have[15]

$$N(s)=\sum_{i=1}^{4} b_i/(s-s_i)\,, \tag{7}$$

$$D(s)=1-\frac{(s-s_0)}{\pi}\int_{(m+1)^2}^{\infty} ds'\frac{(q'^3/s')}{(s'-s)(s'-s_0)} \times\sum_{i=1}^{4}\frac{b_i}{(s'-s_i)}\,. \tag{8}$$

The amplitude $g_{11}(s)$ is now known to the extent of the Balázs residues b_3 and b_4. These residues, however, can be determined by matching the amplitude $g_{11}(s)$, as given by Eqs. (3), (7), and (8), with that calculated from the fixed-energy dispersion relations:

$$A^i(s,t,u)=\frac{1}{\pi}\int_{(m+1)^2}^{\infty} du'\frac{A_u{}^i(u',s)}{u'-u}+\frac{1}{\pi}\int_{4}^{\infty} dt'\frac{A_t{}^i(t',s)}{t'-t}\,,$$

where

$$A^1=A^+,\ A^2=A^-,\ A^3=B^+,\ A^4=B^-. \tag{9}$$

There is a crossed-channel nucleon pole term in $B^{\pm}(s,t,u)$, and a direct-channel one, which in the spirit of Singh and Udgaonkar,[7] can be taken as coming from the high u' and high t' regions in the above integrals. The lower regions of these integrals are presumably exhausted once we take into account N^* in the u-channel absorptive parts and the ρ meson in the t-channel absorptive parts. Projecting out the relevant partial waves from $A^{\pm}(s,t,u)$ and $B^{\pm}(s,t,u)$ and substituting the results in Eq. (2), we obtain

$$g_{11}(s)=g_{11}{}^{(N)}(s)+g_{11}{}^{(N^*)}(s)+g_{11}{}^{(\rho)}(s)\,, \tag{10}$$

where

$$g_{11}{}^{(N)}(s)=-\frac{2g^2s^2}{[(s^{1/2}+m)^2-1][(s^{1/2}-m)^2-1]}\left\{\frac{(s^{1/2}-m)}{[(s^{1/2}-m)^2-1]}Q_1\left(1+\frac{2s(2+m^2-s)}{[(s^{1/2}+m)^2-1][(s^{1/2}-m)^2-1]}\right)\right.$$
$$\left.+\frac{(s^{1/2}+m)}{[(s^{1/2}+m)^2-1]}Q_0\left(1+\frac{2s(2+m^2-s)}{[(s^{1/2}+m)^2-1][(s^{1/2}-m)^2-1]}\right)\right\}-\frac{3g^2s^2(s^{1/2}+m)}{[(s^{1/2}+m)^2-1](s-m^2)}\,, \tag{11}$$

$$g_{11}{}^{(N^*)}(s)=\frac{32W_R{}^2q_R{}^2\gamma_{33}s^2}{3[(s^{1/2}+m)^2-1][(s^{1/2}-m)^2-1]}\left\{-\frac{Q_1(x_R)}{[(s^{1/2}-m)^2-1]}\left(\frac{(W_R-s^{1/2}+2m)3x^x}{(W_R+m)^2-1}+\frac{(W_R+s^{1/2}-2m)}{(W_R-m)^2-1}\right)\right.$$
$$\left.+\frac{Q_0(x_R)}{[(s^{1/2}+m)^2-1]}\left(\frac{(W_R+s^{1/2}+2m)3x^x}{(W_R+m)^2-1}+\frac{(W_R-s^{1/2}-2m)}{(W_R-m)^2-1}\right)\right\}\,, \tag{12}$$

$$g_{11}{}^{(\rho)}(s)=-\frac{12s^2}{\pi[(s^{1/2}+m)^2-1][(s^{1/2}-m)^2-1]}\left\{\frac{1}{[(s^{1/2}-m)^2-1]}Q_1\left(1+\frac{2st_R}{[(s^{1/2}+m)^2-1][(s^{1/2}-m)^2-1]}\right)\right.$$
$$\times[(s^{1/2}-m)(\gamma_1+2\gamma_2)-(\gamma_2/m)(s+\tfrac{1}{2}t_R-m^2-1)]$$
$$+\frac{1}{[(s^{1/2}+m)^2-1]}Q_0\left(1+\frac{2st_R}{[(s^{1/2}+m)^2-1][(s^{1/2}-m)^2-1]}\right)$$
$$\left.\times[(s^{1/2}+m)(\gamma_1+2\gamma_2)+(\gamma_2/m)(s+\tfrac{1}{2}t_R-m^2-1)]\right\}\,, \tag{13}$$

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[13] The kinematical singularities that the amplitude now has can, in principle, be incorporated in the N function; we shall, however, ignore them, as the discontinuities arising from them are not needed in our calculation.

[14] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).

[15] Since the nucleon bound state pole in the calculated N/D amplitude occurs at $s=s_2=m^2$, some caution is needed in handling the term in the N-function that arises from the crossed nucleon "pseudopole." One way out is to replace s_2 by $s_2-\epsilon$, while doing the

with the abbreviations,

$$x_R = 1 + \frac{2s(2+2m^2-s-W_R^2)}{[(s^{1/2}+m)^2-1][(s^{1/2}-m)^2-1]},$$

$$x^x = 1 + \frac{2W_R^2(2m^2+2-s-W_R^2)}{W_R^4-2W_R^2(m^2+1)+(m^2-1)^2},$$

and

$$t_R = m_\rho^2 = 30, \gamma_{33} = 0.12, \gamma_1 = -4.91 \quad \text{and} \quad \gamma_2 = -11.7.$$

[We have chosen these numerical values to be the same as in Ref. 7.]

The amplitude and its derivative as given by Eqs. (3), (7), and (8) can now be matched with that given by Eqs. (10)–(13) and it corresponding derivative. The two Balázs residues can thus be determined. Consequently, the N/D amplitude is now completely known. This amplitude yields a bound state at $s=s_P$, i.e., we have

$$D(s=s_P)=0. \tag{14}$$

The residue of the pole in $g_{11}(s)$, corresponding to this bound state, is essentially the $\pi-N$ coupling constant. This residue can be calculated, and expressing the coupling constant in terms of it, we have

$$g^2 = -\frac{(4s_P-1)}{6(s_P)^{3/2}} \frac{N(s_P)}{D'(s_P)}. \tag{15}$$

The requirement that the calculated nucleon mass $\sqrt{s_P}$ and the coupling constant g^2, as given by Eq. (15), equal the nucleon mass and the $\pi-N$ coupling constant which we fed in through Eqs. (10)–(13),[16] is the self-consistency aspect of the problem.

With the matching point at $s=33.64$,[17] we achieve self-consistency for a mass $\sqrt{s_P}\approx 6.3$ and $g^2\approx 12$. The situation is depicted in Figs. 2 and 3. We have also obtained the low-energy phase shifts (Fig. 4) and the quantity $q^3 \cot\delta_{11}$ (Fig. 5) from our self-consistently calculated amplitude. The scattering length that we obtain for our calculated threshold is $a_{11}\approx -0.16$, and that for the physical threshold is $a_{11}\approx -0.086$. In either case, the calculated scattering length is in good qualitative agreement with the experimentally known value $a_{11}=-0.104\pm 0.006$.[18]

It is worth noting that if one is *not* interested in calculating the nucleon pole parameters but only the $P_{1/2}$, $T=\frac{1}{2}$, $\pi-N$ phase shifts, the Balázs residues can themselves be calculated by demanding that the N/D amplitude have the nucleon pole with the correct

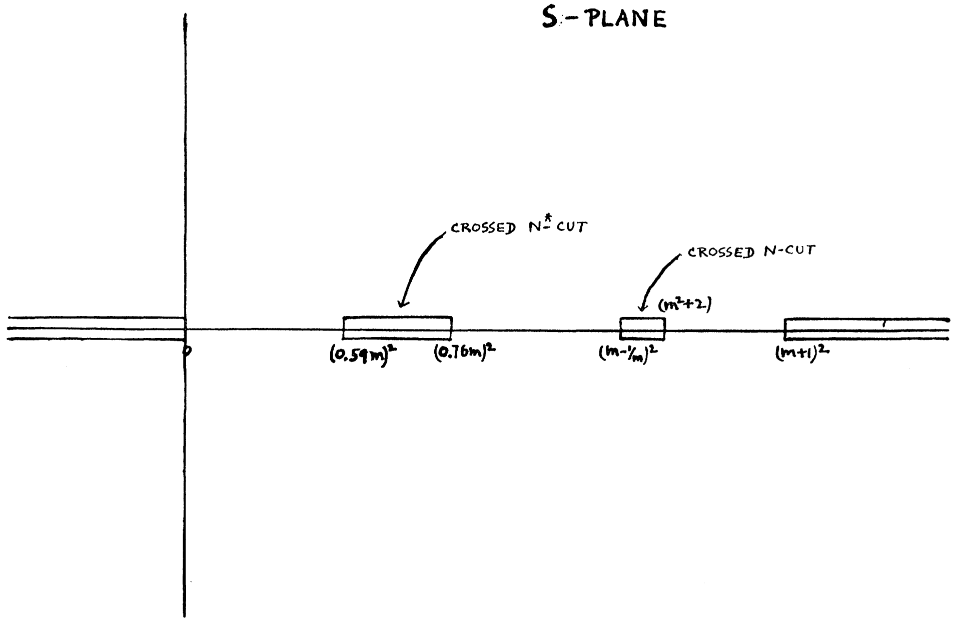

FIG. 1. Approximate singularity structure of the partial-wave pion-nucleon scattering amplitude.

calculation (as is done in Ref. 6) and finally let $\epsilon\rightarrow 0$. We found it more convenient to use the form $D(s)=1-(s-s_0)/(s_2-s_0)$ near $s=s_2=m^2$, so that $b_2/(s-s_2)=\frac{2}{3}f^2m^3D(m^2)/(s-s_2)$ could be written as $\frac{2}{3}f^2m^3/(s_0-s_2)$ and, in this form, this term did not cause any trouble. Near $s=s_2$, the above ansatz for $D(s)$ was always found to be a very good approximation for the calculated $D(s)$.

[16] Actually, in the present problem these quantities enter also through the crossed-channel nucleon contribution to the N function. Further, as the nucleon is one of the incoming (and outgoing) particles, its mass appears at a few more places, namely, in the N^* exchange contribution to the N function, in the Balázs pole terms, and in defining the threshold itself.

[17] We have chosen the subtraction point to be the same as the matching point.

[18] W. S. Woolcock, *Proceedings of Aix-en-Provence International Conference on Elementary Particles* (Centre d'Études Nucléaires de Saclay, Seine et Oise, 1961), p. 459.

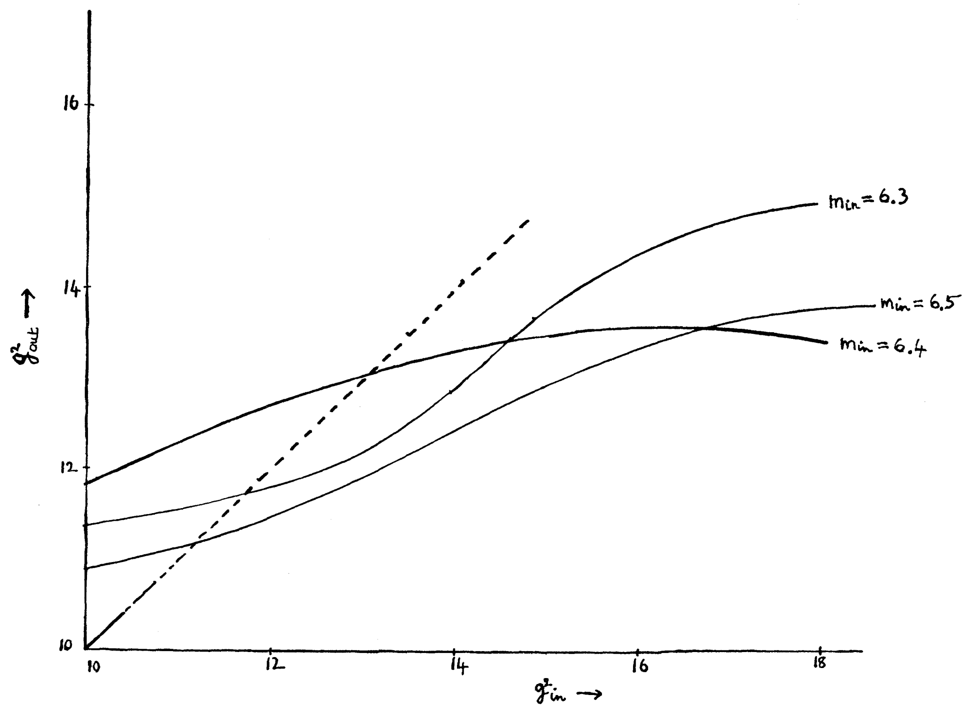

FIG. 2. Variation of the output coupling constant with input value, for various values of input nucleon mass m_{in}. The dashed line corresponds to $g_{\text{in}}^2 \approx g_{\text{out}}^2$.

parameters. This determines the amplitude completely and one now also has the correct threshold. Such an approach was tried by Balázs.[19] He, however, did not include the shortcut due to the crossed-channel N^* in his N function. A recalculation of the $P_{1/2}$, $T=\frac{1}{2}$ phase shifts with his method, but with this additional cut in

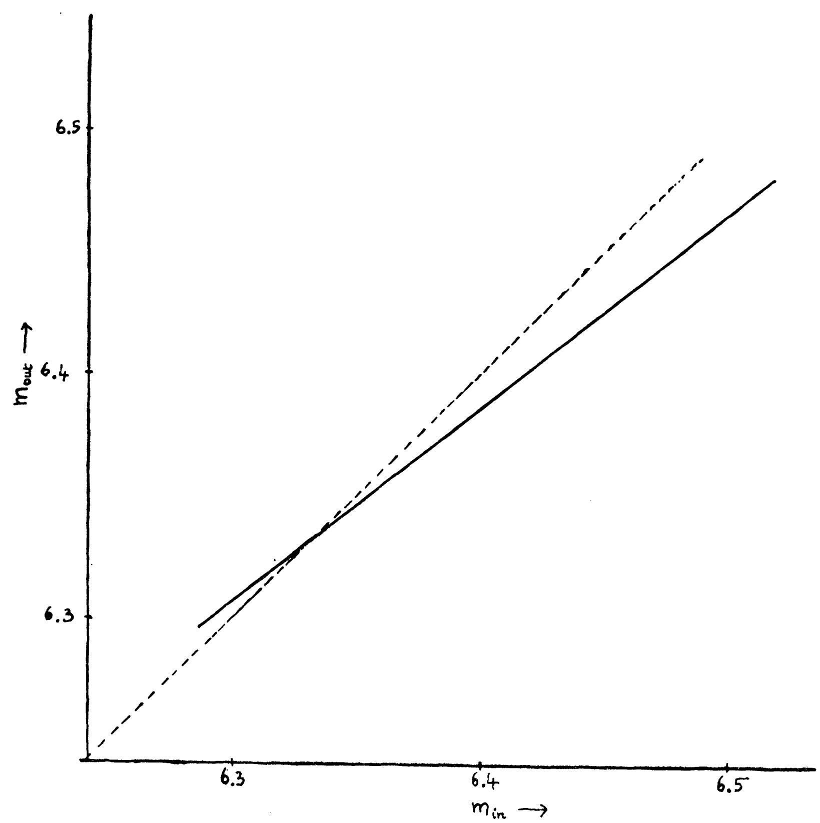

FIG. 3. Graph showing m_{in} versus m_{out}. For any given m_{in}, m_{out} was calculated by using the coupling constant $g_{\text{in}}^2 = g_{\text{out}}^2 = g^2$, obtained from Fig. 2. Note that $m_{\text{in}} = m_{\text{out}} \approx 6.3$, for which Fig. 2 gives $g_{\text{in}}^2 = g_{\text{out}}^2 \approx 12$.

[19] L. A. P. Balázs, Phys. Rev. 128, 1935 (1962).

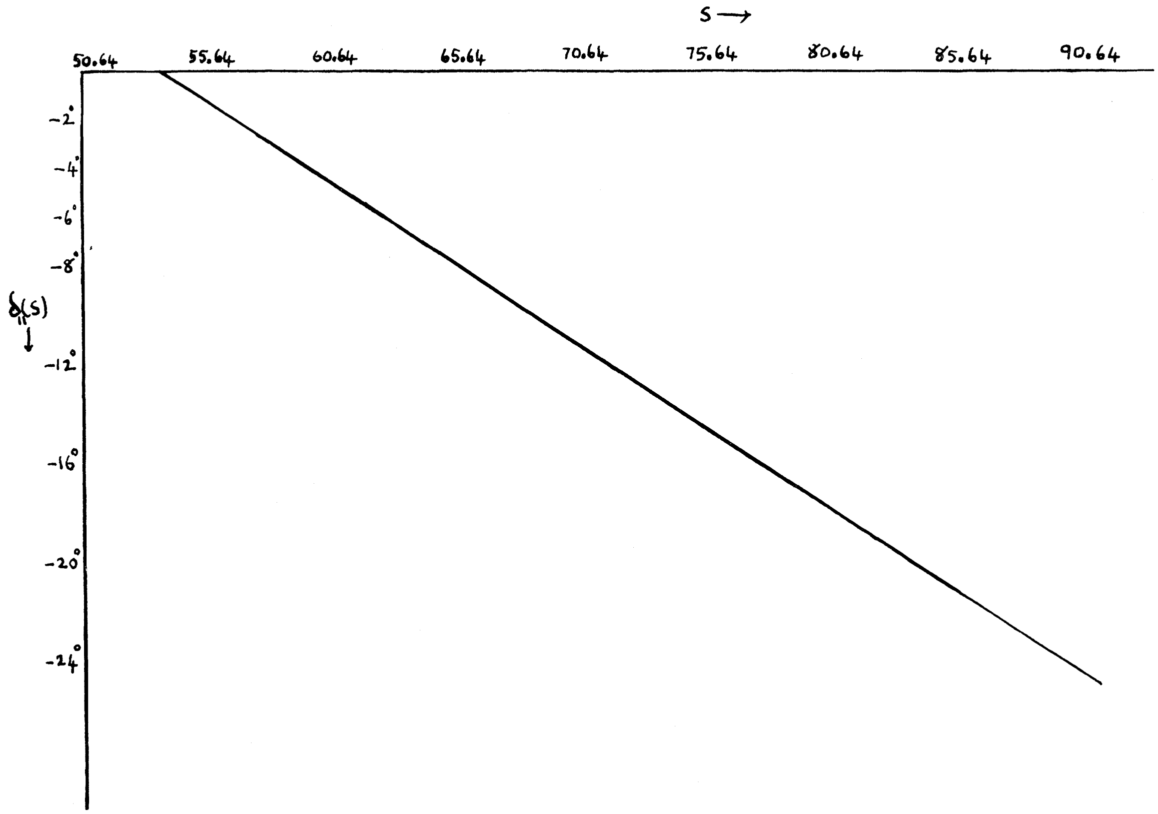

FIG. 4. $P_{1/2}$, $T=\frac{1}{2}$ pion-nucleon phase shifts plotted against the c.m. energy squared, s (expressed in pion mass units).

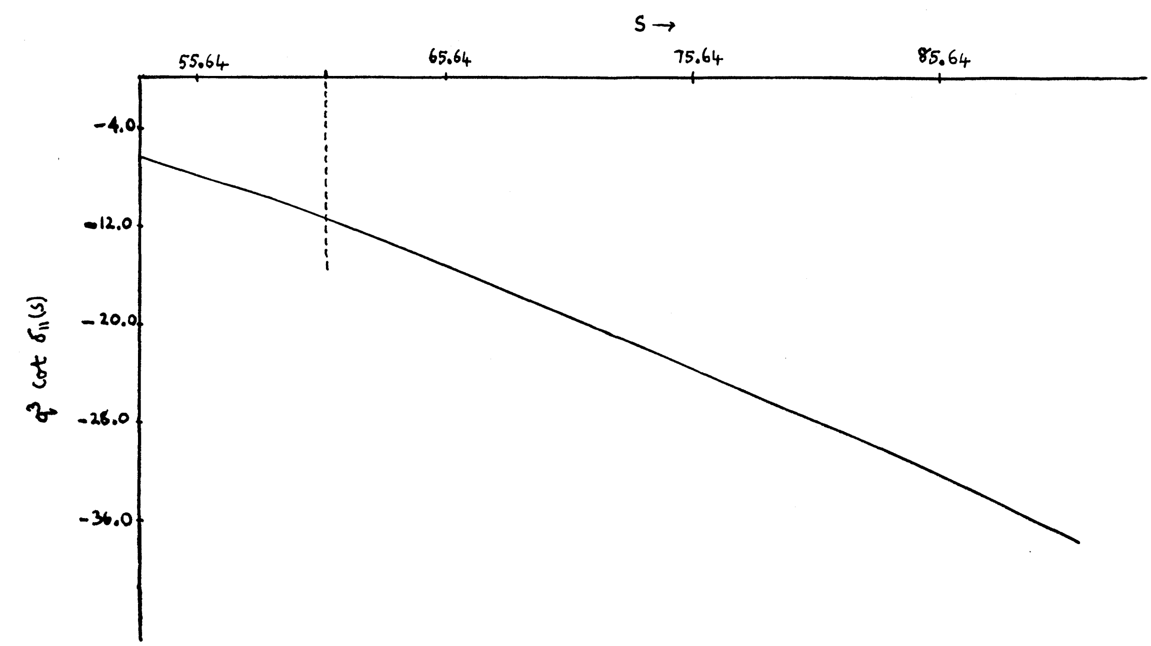

FIG. 5. $q^3 \cot\delta_{11}(s)$ versus s. The dashed line corresponds to the actual threshold, $(7.8)^2$.

the N function, would presumably give a better result.[20,21]

In the present calculation, we tried to see whether the D function (to be precise, the real part of D) shows any other zero in the region above threshold, which could be identified with a possible $P_{1/2}$, $T=\frac{1}{2}$, $\pi-N$ reso-

[20] B. M. Udgaonkar (private communication).

[21] N. Khuri and B. M. Udgaonkar [Phys. Rev. Letters **10**, 172 (1963)] have calculated these phase shifts in a different approach, in which the nucleon is assumed to be a Regge pole.

nance.[22] The answer was in the negative. The D function simply becomes more and more negative after passing through zero corresponding to the bound state (nucleon). This behavior persists even when one adds the contribution of the possible $P_{1/2}$, $T=\frac{1}{2}$ resonance with a mass around 1485 MeV,[22] and the width within a wide range of possibility, in the amplitude $g_{11}(s)$ in Eq. (10). This contribution is simply

$$g_{11}{}^{(P)}(s)=g_{11}{}^{(P)H}(s)+g_{11}{}^{(P)L}(s)\,, \tag{16}$$

with

$$g_{11}{}^{(P)H}(s)=\frac{W_P{}^2\gamma_P[(s^{1/2}-m)^2-1]}{[(W_P-m)^2-1](W_P-W)} \tag{17}$$

and

$$g_{11}{}^{(P)L}(s)=-\frac{W_P{}^2q^2{}_P\gamma_P}{6q^4[(W_P-m)^2-1]} \times\{[(s^{1/2}+m)^2-1](W_P+s^{1/2}-2m)Q_1(x_P) +[(s^{1/2}-m)^2-1](s^{1/2}+2m-W_P)Q_0(x_P)\}\,, \tag{18}$$

where γ_P is the reduced half-width of the resonance, and

$$x_P=1+\frac{2s(2+2m^2-s-W_P{}^2)}{[(s^{1/2}+m)^2-1][(s^{1/2}-m)^2-1]}.$$

If there *really* is a resonance present, it would be difficult to produce it in the present approach, as one does not know how one could manage a D function with the required behavior.

III. VARIATION OF MATCHING POINT

In this section, we would like to comment briefly on the situation with regard to the change in the matching point. It should be noted that the presence of the short N^* cut and the short nucleon cut, both of which lie between the left-hand cut and the physical cut,[23] restricts the region of the real s accessible for matching purposes considerably, as the matching point should be as far away from *all* singularities as is possible. Another point to be noted is that the end points of both the short cuts and the threshold (the branch point for the physical cut) are all functions of the nucleon mass. The positions of all these points consequently change as we vary the input nucleon mass. Many points which could be used for matching do not remain good as we vary the input mass in a range within which self-consistency is expected.

The matching point used in the calculations of the last section is a rather "safe" one, in addition to being the most convenient from the calculational point of view. The points close to this one are also relatively safe. We tried, for instance, the point $s=s_0=28.0$. The results did show variation: the mass was close to 6.0 but the coupling constant changed by almost 50%. We also tried points $s=s_0=45.0$. This point is quite close to, in fact slightly above, the point where self-consistency is expected, i.e., the point where the N/D amplitude has the nucleon pole. In this region for matching, the Balázs residues fluctuate wildly as we vary the input mass. The reason is that $g_{11}{}^{(N)}(s=s_0)$ has the denominator (s_0-m^2), which makes $g_{11}{}^{(N)}(s)$ change sign through infinity as we vary the input from $m^2<s_0$ to $m^2>s_0$. Although one can get results in this region, they cannot be taken seriously. That a situation of this sort occurs in calculations of bound states (and not of resonances, as resonances occur above thresholds where one does not match anyway) has been noted earlier also.[24]

In conclusion, it should be said that although the region in which one could move the matching point is not really arbitrarily large, yet the final results are *not quite stable* against whatever variation in the position of the matching point one is legitimately allowed to try. The reason is, presumably, that the amplitude one calculates to match with is quite approximate. Within all these limitations, however, the present method does seem to be useful in the context of the pion-nucleon problem, as one gets interesting results for all low angular-momentum partial waves by using the same matching point.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge the benefit of stimulating conversations with S. R. Choudhury. We are grateful to Professor R. C. Majumdar for encouragement and for providing facilities that enabled us to complete this work. One of us (L.K.P.) is also grateful to C.S.I.R. for financial support.

[22] L. D. Roper, Phys. Rev. Letters **12**, 340 (1964).

[23] Although in the N function these cuts are approximated by pole terms, they show up explicitly through Eqs. (11) and (12).

[24] S. Rai Choudhury and L. K. Pande, Phys. Rev. **135**, B1027 (1964).